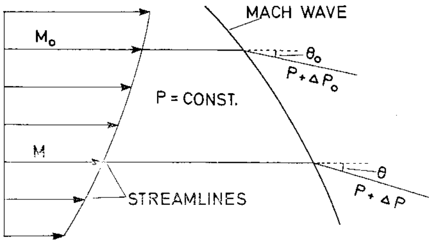

Fig. 1 Refraction of a Mach wave (compression wave) in a supersonic, isobaric flow exhibiting a certain Mach number profile.

compression wave reaches the ground. This procedure requires the numerical evaluation of a series of n products.

Instead of doing such numerical calculations, the problem can also be described by a simple, analytic equation that relates the intensity of the transmitted wave to the local Mach number M of the freestream (Fig. 1). If the intensity of the wave is expressed in terms of the deflection angle θ of each originally horizontal streamline, this relation is

$$(\theta/\theta_0) = (M_0/M)[(M^2 - 1)/(M_0^2 - 1)]^{1/4} \qquad (1)$$

M_0 and θ_0 are initial values. Equation (1) has been reported by several authors[4-6] in different connection.

For the sonic boom problem, however, one is interested in determining the pressure change Δp across the wave. Again under the same assumptions (i.e., with a two-dimensional, parallel, supersonic flow of constant static pressure, but exhibiting a certain Mach number profile), one derives from Eq. (1)

$$(\Delta p/\Delta p_0) = (M/M_0)[(M_0^2 - 1)/(M^2 - 1)]^{1/4} \qquad (2)$$

M_0 and Δp_0 now play the role of initial values.

Δp as a function of M (see Fig. 2) has a minimum for $M = (2)^{1/2}$ and goes to infinity for $M = 1$; Eqs. (1) and (2) are not valid for values of M very close to 1.

With the aid of Fig. 2, the problem described in the beginning of this Note can be solved as follows. The strength of the sonic boom on the ground is determined with one of the well-known methods for a so-called standard atmosphere. This corresponds to one particular point on the curve in Fig. 2 or to a set of initial values M_0, Δp_0. Then, the deviations from the standard atmosphere in the weather layer due to wind (tail-wind or head-wind) and to temperature changes are converted to changes in Mach number. ΔM may be either positive or negative depending on the directions of the temperature gradient and wind. Having the new Mach number $M = M_0 + \Delta M$, one reads directly the corresponding pressure change Δp of the sonic boom.

This procedure, which can be handled much easier than the numerical methods mentioned before, yields the same

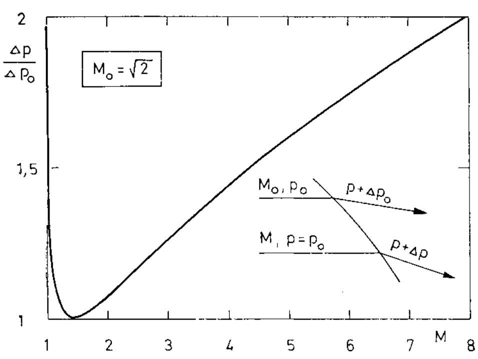

Fig. 2 Pressure change ($\Delta p/\Delta p_0$) according to Eq. (2) with $M_0 = (2)^{1/2}$ as initial value.

quantitative results, e.g., curves such as those in Fig. 4 of Ref. 3. These changes of the sonic boom strength are usually not higher than 6 to 7%, unless one approaches the local Mach number $M = 1$ in the modified reference system. Here, however, one approaches also the limits of validity of the governing equations.

References

[1] Kane, E. J., "Some Effects of the Nonuniform Atmosphere on the Propagation of Sonic Booms," *Journal of the Acoustical Society of America*, Vol. 39, No. 5, May 1966, pp. S26–S30.

[2] Auriol, A. et al., "Réfraction des Ondes de Choc Aériennes. 1ére Partie: Cas d'une Onde de Choc d'Intensité Constante se Réfractant dans une Atmosphère Isobare avec Gradients Verticaux de Température et de Vent," ISL T 14/66, July 1966, Institute Franco-Allemand de Recherches, St. Louis, France.

[3] Pan, Y. S., "Effects of Winds and Inhomogeneous Atmosphere on Sonic Boom," *AIAA Journal*, Vol. 6, No. 7, July 1968, pp. 1393–1395.

[4] Liepmann, H. W., private communication (based on an unpublished report of 1948 by the Transonic Research Group, Guggenheim Aeronautical Lab., California Institute of Technology, Pasadena, Calif.).

[5] Weinbaum, S., "Rapid Expansion of a Supersonic Boundary Layer and Its Application to the Near Wake," *AIAA Journal*, Vol. 4, No. 2, Feb. 1966, pp. 217–226.

[6] Merzkirch, W., "Die Wechselwirkung einer Machschen Welle mit einem anisentropen Überschallströmungsfeld," *Zeitschrift fuer Angewandte Mathematik und Mechanik*, Vol. 48, No. 8, 1968, pp. T216–T218.

Low-Frequency Approximation in Unsteady Aerodynamics

Gunter W. Brune*
The Boeing Company, Seattle, Wash.

Nomenclature

Æ	=	aspect ratio
$f(x_1,y_1)$	=	distribution function of surface singularities
H	=	unit step function
k	=	$\omega l/U_\infty$, reduced frequency
l	=	reference length
M	=	Mach number
S	=	wing surface
U_∞	=	freestream velocity
x,y,z	=	nondimensional coordinates, moving with flight velocity in negative x direction
x_0	=	parameter in x integration
β	=	$\lvert 1 - M^2 \rvert^{1/2}$
ϕ	=	velocity potential, referred to $U_\infty l$
ω	=	frequency

Introduction

Dynamic stability derivatives of airplanes may be computed by applying a simplified unsteady aerodynamic theory. This theory is termed the first order in frequency or low-frequency approximation. The approximation is valid for unsteady motions that are characterized by reduced frequencies that are small by comparison with unity. It is therefore applicable to the problem of predicting the unsteady aerodynamic forces that affect the stability characteristics of large aircraft. The purpose of this Note is to define

Received May 6, 1969. This research was carried out under NASA-Ames Contract NAS 2-5006. The author wishes to acknowledge the helpful advice and criticism of A. R. Dusto.

* Aerodynamicist, Commercial Airplane Division.

the range of valid application in terms of aspect ratio and Mach number.

It is common practice to formulate the theory of oscillating lifting wings by an integral equation relating unsteady downwash and lift distributions. Detailed information about the kernel of the integral equation including frequency expansions was published by Watkins, Runyan, and Woolston[1] in subsonic and by Watkins and Berman[2] in supersonic flow. Their results show that a pure power series expansion in terms of the reduced frequency k is valid only in supersonic flow; whereas, in subsonic flow terms of order $k^2 \ln k$ occur in three-dimensional and those of order $k \ln k$ in the two-dimensional flow problem. Hence, in subsonic flow an approximation retaining only the first order in frequency term certainly is not valid for wings of large aspect ratios with mainly two-dimensional flow pattern, but the approximation may be valid for medium- and low-aspect ratio wings.

Conditions in Subsonic Flow

The desired conditions for a valid first order in frequency approximation can be obtained by investigating the velocity potential of oscillating lifting wings. Assuming harmonic motions, the complex amplitude of the velocity potential is given by the integral

$$\phi(x,y,z;k,M) = -\frac{1}{4\pi}\int_{-\infty}^{x}\iint_S f(x_1,y_1)\frac{\partial}{\partial z} \times \left(\frac{1}{R}\, e^{ik(M/\beta^2)[M(x'-x_1)-R]}\right) dx_1 dy_1 e^{ik(x'-x)}\, dx' \quad (1)$$

with

$$R = [(x'-x_1)^2 + \beta^2(y-y_1)^2 + \beta^2 z^2]^{1/2}$$

This integral is a solution of the linearized unsteady potential equation. The corresponding acceleration potential is the inner double integral of (1) and was derived by integrating an unknown distribution $f(x_1,y_1)$ of pressure doublets on the wing. Equation (1) satisfies the boundary conditions in the wake and at infinity. The reduced frequency k is arbitrary within the limitations of the linearized wing problem.

A straightforward frequency expansion of (1) is doubtful with respect to the lower limit of the outer integral. It becomes necessary to solve the integral approximately and to expand the result for low frequencies. This will be done adapting Garner's analysis.[3]

Integral (1) can be divided into

$$\phi(x,y,z;k,M) = \int_{-\infty}^{x-x_0} \psi(x',y,z;k,M) e^{ik(x'-x)}\, dx' + \int_{x-x_0}^{x} \psi(x',y,z;k,M) e^{ik(x'-x)}\, dx' \quad (2)$$

where the variable ψ represents the surface integral in Eq. (1). The parameter x_0 is chosen such that the first integral, referred to as ϕ_∞, becomes an integration from $-\infty$ to some chord lengths ahead of the wing and the second integral, termed ϕ_W, is bounded to the wing region. Figure 1 shows the different regions of integration.

The potential ϕ_W may be expanded directly to first order in frequency. The conditions for that expansion are

$$k \ll 1,\; k \ll (1-M^2)/M^2,\; k \ll (1-M^2)/M \quad (3)$$

where $|x'-x_1|_{\max}$, $|x'-x|_{\max}$, and $R_{\max}$ are to be of order unity in the wing region.

The potential ϕ_∞ will be solved approximately under the assumptions that

$$\beta^2(y-y_1)^2 + \beta^2 z^2 \ll (x'-x_1)^2$$
$$(x'-x_1)^2 \approx (x'-x)^2 \quad (4)$$

in the region ahead of the wing. Performing the indicated

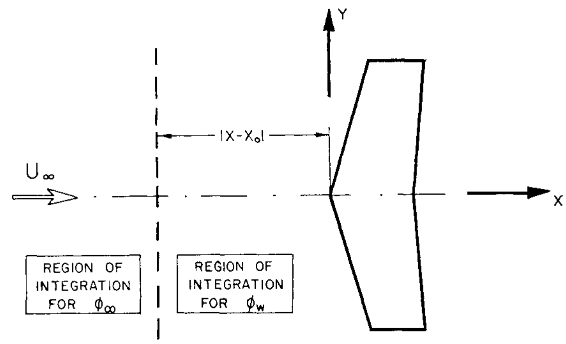

Fig. 1 Wing in subsonic flow.

differentiation in integral (1) and introducing the assumptions (4), the potential ϕ_∞ becomes

$$\phi_\infty(x,y,z;k,M) \approx \frac{z}{4\pi}\iint_S f(x_1,y_1)dx_1dy_1 \times \int_{-\infty}^{x-x_0} \frac{ikM(x-x')+\beta^2}{(x-x')^3}\, e^{-i[k/(1-M)](x-x')}\, dx' \quad (5)$$

The solution of the last integral in Eq. (5) for low frequencies is

$$\int_{-\infty}^{x-x_0} \frac{ikM(x-x')+\beta^2}{(x-x')^3}\, e^{-i[k/(1-M)](x-x')}\, dx' = \frac{\beta^2}{2x_0^2} - i\,\frac{k}{x_0} + \frac{1}{2}k^2\ln k + 0(k^2) \quad (6)$$

That series is valid to first order in frequency if

$$k|\ln k| \ll 2/x_0 \quad (7)$$

The parameter x_0 can be removed from inequality (7) by modifying the assumptions (4), which also may be written as

$$\beta^2(y-y_1)^2_{\max} \ll (x'-x)^2_{\min} \quad (8)$$

neglecting $\beta^2 z^2$. The term $|y-y_1|_{\max}$ is of the same order of magnitude as wing span; therefore, it is equivalent to the aspect ratio $\mathcal{R}$ if all lengths are referred to the mean wing chord. In the considered region, $|x'-x|_{\min}$ is identified as x_0 so that the inequality (8) becomes

$$\beta\mathcal{R} \ll x_0 \quad (9)$$

Combining (7) and (9) yields another condition for a valid first order in frequency approximation in subsonic flow;

$$k|\ln k| \ll 2/\beta\mathcal{R} \quad (10)$$

Conditions in Supersonic Flow

The integral of the velocity potential of harmonically oscillating wings in supersonic flow reads

$$\phi(x,y,z;k,M) = -\frac{1}{2\pi}\int_{-\infty}^{x}\iint_S f(x_1,y_1)\frac{\partial}{\partial z} \times \left[\frac{H\{(x'-x_1)-\beta[(y-y_1)^2+z^2]^{1/2}\}}{R}\, e^{-ik(M^2/\beta^2)(x'-x_1)} \times \cos\left(k\,\frac{M}{\beta^2}R\right)\right] dx_1dy_1 e^{-ik(x'-x)}\, dx' \quad (11)$$

with $R = [(x'-x_1)^2 - \beta^2(y-y_1)^2 - \beta^2 z^2]^{1/2}$. The inner double integral is restricted to that part of the wing surface S within the forecone of the field point x, y, z. As indicated by the unit step function H, the perturbations vanish before the Mach cone from the foremost point of the wing. Therefore, the outer integration is actually restricted to the wing region. An expansion of the velocity potential (11) for low frequencies becomes quite straightforward. Assuming that

$|x' - x|_{\max}$, $|x' - x_1|_{\max}$, and $R_{\max}$ are of order unity, the conditions for a valid first order in frequency approximation in supersonic flow read

$$k \ll 1, k \ll (M^2 - 1)/M^2 \qquad (12)$$

Conclusions

Conditions for a valid first order in frequency approximation of an unsteady lifting wing theory were derived. In both subsonic and supersonic flow the frequency is restricted beyond the low-frequency assumption $k \ll 1$. In subsonic flow the frequency is seriously restricted by the slenderness factor $\beta Æ$ of the wing. The condition

$$k|\ln k| \ll 2/\beta Æ$$

indicates that for small but finite reduced frequencies k the factor $\beta Æ$ must be equal to or smaller than order of unity to render a consistent first-order theory. For extremely low frequencies, the unsteady problem would degenerate into a quasi-steady problem no longer calling for an unsteady solution. The condition excludes the application to large-aspect ratio wings, and those of medium-aspect ratios are only covered at higher subsonic Mach numbers. A similar limitation due to aspect ratio does not exist in supersonic flow. There are additional conditions in subsonic and supersonic flow that restrict the frequency with respect to the Mach number M. The inequalities

$$k \ll (1 - M^2)/M \quad (M < 1)$$

$$k \ll (M^2 - 1)/M^2 \quad (M > 1)$$

show that the first order in frequency theory is not applicable in a wide transonic flow region.

References

[1] Watkins, C. E., Runyan, H. L., and Woolston, D. S., "On the Kernel Function of the Integral Equation Relating the Lift and Downwash Distributions of Oscillating Finite Wings in Subsonic Flow," Rept. 1234, 1955, NACA.

[2] Watkins, C. E. and Berman, J. H., "On the Kernel Function of the Integral Equation Relating Lift and Downwash Distributions of Oscillating Wings in Supersonic Flow," Rept. 1257, 1956, NACA.

[3] Garner, H. C., "Multhopp's Subsonic Lifting-Surface Theory of Wings in Slow Pitching Oscillations," ARC TR R&M 2885, 1956, Aeronautical Research Council.